Section 3.2

The Derivative as a Function

(1) The Derivative Function

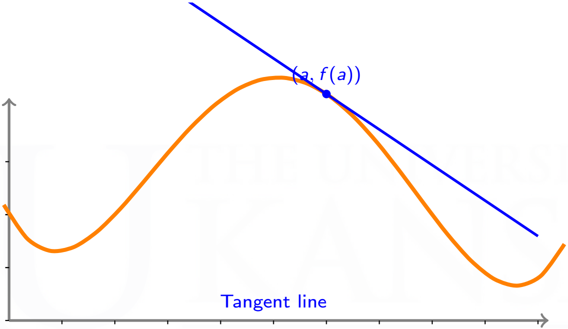
(2) Notation of Derivatives

(3) Derivative Rules

1. Power Rule
2. Constant Multiple Rule
3. Sum/Difference Rules
4. Derivative of the Natural Exponential Function

(4) The Definition of e

(5) Revisiting Some Examples Using The Rules



Derivative of a Function at a Point

The **derivative** of a function $y = f(x)$ at $x = a$ is (if it exists)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For example: The derivative function of $f(x) = x^2 + 4$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left((x+h)^2 + 4\right) - \left(x^2 + 4\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(x^2 + 2xh + h^2 + 4\right) - \left(x^2 + 4\right)}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h = 2x$$

Notations for the Derivative

Let $y = f(x)$ be a differentiable function.

	Derivative Function	Derivative at $x = a$
Lagrange	$y'(x) = f'(x)$	$f'(a)$
Leibniz	$\frac{d}{dx}(f(x)) = \frac{dy}{dx} = \frac{df}{dx}$	$\frac{d}{dx}(f(x))\Big\vert_{x=a} = \frac{dy}{dx}\Big\vert_{x=a}$

Higher Derivatives:

- $f''(x), f'''(x), \ldots, f^{(n)}(x), \ldots$
- $\frac{d}{dx}\left(\frac{d}{dx}(f(x))\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ $\qquad$ $\left(\frac{d}{dx}\right)^n (f(x)) = \frac{d^ny}{dx^n}$

Derivatives and Motion

A particle moves along a straight line. The function $s(t)$ represents its distance from a fixed point at time t.

- The **velocity** of the object at time t is $v(t) = s'(t) = \dfrac{ds}{dt}$
- The **acceleration** at time t is $a(t)$:

$$a(t) = s''(t) = v'(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt}$$

- The **jerk** at time t is $j(t)$:

$$j(t) = s'''(t) = v''(t) = a'(t) = \frac{d^3s}{dt^3} = \frac{d^2v}{dt^2} = \frac{da}{dt}$$

Units

$s(t)$ distance , $v(t)\ \dfrac{\text{distance}}{\text{time}}$, $a(t)\ \dfrac{\text{distance}}{\text{time}^2}$, $j(t)\ \dfrac{\text{distance}}{\text{time}^3}$

Derivative Rules

Suppose that f and g are differentiable functions and c is a constant.

Derivative of Constants

$$\frac{d}{dx}(c) = 0$$

Constant Multiple Rule

$$\frac{d}{dx}(cf(x)) = c\,\frac{d}{dx}(f(x))$$

Sum and Difference Rules

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

The Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Differentiating polynomials is now simple! $\quad \dfrac{d}{dx}\left(x^7 - 12x^4 + x^2 - x - 3\right)$

$$= \frac{d}{dx}\left(x^7\right) - 12\frac{d}{dx}\left(x^4\right) + \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(x) + \frac{d}{dx}(-3) = 7x^6 - 48x^3 + 2x - 1$$

Exponential Functions and their Derivatives

Recall that an **exponential function** is a function of the form $f(x) = b^x$, where b is a positive constant. Can we determine their derivatives?

$$f'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} = b^x \boxed{\lim_{h \to 0} \frac{b^h - 1}{h}}$$

The derivative depends on the value of $\displaystyle\lim_{h \to 0} \frac{b^h - 1}{h} \boldsymbol{= f'(0).}$

It will take some time, but we eventually will show that

$$\lim_{h \to 0} \frac{b^h - 1}{h} = \ln(b).$$

Exponential Functions and their Derivatives

Derivatives of Exponential Functions

For all positive real numbers b,

$$\frac{d}{dx}\left(b^x\right) \;=\; b^x \left(\lim_{h\to 0} \frac{b^h - 1}{h} \right) \;=\; b^x \ln(b)$$

Euler's constant $\boldsymbol{e}$ is the unique number satisfying the equation

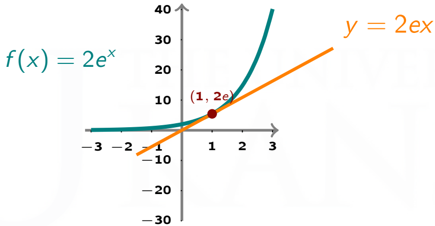
$$\ln(e) = \lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

(It so happens that $e \approx 2.71828184590452353602874713527\ldots$)

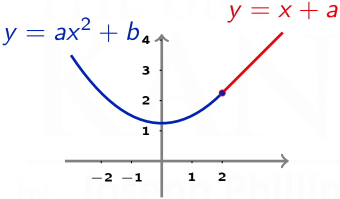
Derivative of e^x

$$\frac{d}{dx}\left(e^x\right) = e^x$$

Example I: Find an equation of the tangent line to $f(x) = 2e^x$ at $x = 1$.



Example II: Let $f(x) = \begin{cases} ax^2 + b & x \leq 2 \\ x + a & x > 2 \end{cases}$. Find the values for **a** and **b** which make the function differentiable.



Example III: Let $p(x) = x^2 + ax + b$. Find values for **a** and **b** which make the graph of $p(x)$ pass through $(-1, -3)$ and make the tangent line to p at $x = -1$ horizontal.